The embedding order is too weak, but it is a good starting point for the development of suitable reduction relations. ⇒ Special notion for reduction orders that contain the embedding order.

Def 4.4.1 (Simplification Order (Dershowitz, 1987))
A reduction order $\succ$ where $s \succ t$ holds whenever $s \succ_{emb} t$, is called a simplification order.

If an order contains the embedding order, then there is a "simple" way to prove its well-foundedness: one only has to check that the order is irreflexive ($t \not\succ t$ for all t).
Reason: Kruskal's Theorem.

Thm 4.4.2 (Kruskal's Theorem)
As usual, the signature Σ is finite.
(a) For any infinite sequence of ground terms $t_0, t_1, t_2, \ldots$ there exist $i, j \in \mathbb{N}$ with $i < j$ such that $t_i \preceq_{emb} t_j$.

$minus(0,0)$, $succ(0)$, 0, $minus(s(0),0)$, ...
is $\succ$

(b) Every stable monotonic transitive

minus(0,0), succ(0), 0, minus(s(0),0), ...

is $\succeq_{emb}$ greater than the first term

(b) Every stable, monotonic, transitive relation $\succ$ that satisfies the <u>subterm property</u> ($f(x_1, \ldots, x_n) \succ x_i$ for all $1 \leq i \leq n$) contains the embedding order (i.e., $s \succ_{emb} t \Rightarrow s \succ t$).

(c) Every stable, monotonic, transitive, <u>irreflexive</u> relation that satisfies the subterm property is well founded. Thus, it is a simplification order.

<u>Proof</u>: (a) see literature

(b) To show: $s \succ_{emb} t$ implies $s \succ t$
(Simple proof by structural induction on s.)

(c) $\succ$: stable, monotonic, transitive, contains $\succ_{emb}$, irreflexive.

To show: $\succ$ is well founded.

Assume that there is an infinite sequence

$$t_0 \succ t_1 \succ t_2 \succ \ldots$$

Let σ replace all variables in $t_0, t_1, \ldots$ by ground terms. Then stability of $\succ$ implies:

$$t_0\sigma \succ t_1\sigma \succ t_2\sigma \succ \ldots$$

By Kruskal's Theorem (a) there exist i, j with $i < j$ such that

$$t_i\sigma \preceq_{emb} t_j\sigma \,.$$

Since $\succ$ contains $\succ_{emb}$, we also have: $t_i\sigma \preceq t_j\sigma$.

Thus: $t_i\sigma \succ \ldots \succ t_j\sigma \succeq t_i\sigma$

By transitivity of $\succ$: $t_i\sigma \succ t_i\sigma$

which contradicts irreflexivity of $\succ$. ☒

Now: define suitable relations and prove that they satisfy stability, monotonicity, transitivity, irreflexivity, subterm property.

Idea: Take the embedding order and improve its two main drawbacks:

- weak comparison of terms s and t if $s = f(\ldots)$, $t = g(\ldots)$
- weak comparison of terms s and t if $s = f(\ldots)$, $t = f(\ldots)$. Here, one needs better ways to compare <u>tuples of terms</u>.

There are two main ways to compare tuples:

- lexicographically or ← lexicographic path order
- as multisets ← recursive path order

Lexicographic combination of two relations allows us to compare tuples of objects.

<u>Def 443</u> (Lexicographic Combination of Relations)

Let $\succ_1$ be a relation on a set T_1 and $\succ_2$ be a relation

Let $>_1$ be a relation on a set T_1 and $>_2$ be a relation on a set T_2 (i.e., $>_i \subseteq T_i \times T_i$)

Then the <u>lexicographic combination</u> $>_{1\times 2}$ is a relation on $T_1 \times T_2$, which is defined as follows:

$$(s_1, s_2) >_{1\times 2} (t_1, t_2) \text{ iff } s_1 >_1 t_1 \text{ or } (s_1 = t_1 \text{ and } s_2 >_2 t_2)$$

Similarly, one can define the lexicographic combination of arbitrary many relations $>_1, \dots, >_n$:

$(s_1, \dots, s_n) >_{1\times \dots \times n} (t_1, \dots, t_n)$ iff there exists an $i \in \{1, \dots, n\}$ with $s_i >_i t_i$ and $s_j = t_j$ for all $1 \le j < i$.

The n-fold lexicographic combination of a relation $>$ with itself is denoted $>^n_{lex}$. (This is the case where $> = >_1 = \dots = >_n$).

<u>Ex. 4.4.4.</u> $(3,5) \; (>_{\mathbb{N}})^2_{lex} \; (2,6) \; (>_{\mathbb{N}})^2_{lex} \; (2,5)$

Words in a lexicon are also ordered lexicographically. Let $>_{alph}$ be the order of letters in the alphabet, i.e., $a >_{alph} b >_{alph} \dots >_{alph} z$.

hans $(>_{alph})^4_{lex}$ hugo $(>_{alph})^4_{lex}$ juli.

Well-foundedness is maintained under lexico-

graphic combinations.
But the order in a lexicon is not well founded:

$$a > ba > bba > bbba > \dots$$

Reason: Here, the length of the tuples is not bounded.

<u>Thm 4.45</u> (Well-Foundedness of Lexicographic Combinations)

Let $\succ_1$ be a relation on $T_1 \neq \emptyset$ and $\succ_2$ be a relation on $T_2 \neq \emptyset$. Then $\succ_1$ and $\succ_2$ are well founded iff their lexicographic combination $\succ_{1\times 2}$ is well founded.

<u>Proof</u>: "$\Leftarrow$": Let $\succ_{1\times 2}$ be well founded.

If $\succ_1$ were not well founded, then there would exist a sequence $\mu_0 \succ_1 \mu_1 \succ_1 \mu_2 \succ_1 \mu_3 \succ_1 \dots$

Let $v \in T_2$. Then $(\mu_0, v) \succ_{1\times 2} (\mu_1, v) \succ_{1\times 2} (\mu_2, v) \succ_{1\times 2} \dots$ ↯

Similarly, if $\succ_2$ were not well founded, then there would exist a sequence $v_0 \succ_2 v_1 \succ_2 \dots$.

Let $\mu \in T_1$. Then $(\mu, v_0) \succ_{1\times 2} (\mu, v_1) \succ_{1\times 2} \dots$ ↯

"$\Rightarrow$": Assume that $\succ_{1\times 2}$ were not well founded:

$$(\mu_0, v_0) \succ_{1\times 2} (\mu_1, v_1) \succ_{1\times 2} (\mu_2, v_2) \succ_{1\times 2} \dots$$

Thus: $\mu_0 \succeq_1 \mu_1 \succeq_1 \mu_2 \succeq_1 \dots$

Since $\succ_1$ is well founded, there is an $i \in \mathbb{N}$ such that

$u_i = u_{i+1} = u_{i+2} = \dots$

Thus: $v_i >_2 v_{i+1} >_2 v_{i+2} >_2 \dots$

This contradicts well-foundedness of $>_2$. $\boxtimes$

Now we define the <u>lexicographic path order</u>, which is a more powerful simplification order than the embedding order.

- lpo should again contain the subterm relation $\rhd$ $\Rightarrow$ first condition for lpo is the same as for $>_{emb}$
- $>_{emb}$ is weak when comparing terms $f(s_1,\dots,s_n)$ and $g(t_1,\dots,t_m)$

E.g: $plus(succ(x),y)) \not>_{emb} succ(plus(x,y))$

Solution: assign different weights to function symbols

We use an order $\sqsupset$ on function symbols (precedence).

If $f \sqsupset g$, then $f(s_1,\dots,s_n) >_{lpo} g(t_1,\dots,t_m)$.

To make $>_{lpo}$ well founded, $\sqsupset$ must also be well founded.

If $f \sqsupset g$, then: $f(x) >_{lpo} g(f(x))$ ← This must not be allowed. Otherwise, $>_{lpo}$ would not be well founded.

By first condition: $g(f(x)) >_{lpo} f(x)$

Solution:

$f(s_1,\dots,s_n) >_{lpo} g(t_1,\dots,t_m)$

if $f \sqsupset g$ and

$$f(s_1,\dots,s_n) \succ_{lpo} t_1, \dots, f(s_1,\dots,s_n) \succ_{lpo} t_m$$

- $\succ_{emb}$ is also weak when comparing terms that start with the same fct. symbol:

$f(s_1,\dots,s_n) \succ_{lpo} f(t_1,\dots,t_n)$ if

$$(s_1,\dots,s_n) \ (\succ_{lpo})^n_{lex} \ (t_1,\dots,t_n)$$

This means: $s_1 = t_1,\ s_2 = t_2, \dots, s_{i-1} = t_{i-1},\ s_i \succ_{lpo} t_i$

Is this enough to guarantee well-foundedness of lpo?

$$f(succ(0), 0) \succ_{lpo} f(0, f(succ(0), 0))$$ ← This must not hold,

since $succ(0) \succ_{lpo} 0$

since $f(0, f(s(0), 0)) \succ_{lpo} f(s(0), 0)$

i.e., $\succ_{lpo}$ would not be well founded

To prevent this, we define:

$f(s_1,\dots,s_n) \succ_{lpo} f(t_1,\dots,t_n)$ if

$$s_1 = t_1, \dots, s_{i-1} = t_{i-1},\ s_i \succ_{lpo} t_i,$$
$$f(s_1,\dots,s_n) \succ_{lpo} t_{i+1}, \dots, f(s_1,\dots,s_n) \succ_{lpo} t_n$$

Def 4.4.6 (Lexicographic Path Order, Kamin+Levy 1980)

- see slide -

We will show that the LPO is a simplification order $\Rightarrow$ LPO can be used for termination proofs of TRSs.

<u>Ex 4.47</u>

$plus(0, y) \to y$

$plus(s(x), y) \to s(plus(x, y))$

$times(0, y) \to 0$

$times(s(x), y) \to plus(y, times(x, y))$

Rules 1 and 3 are decreasing w.r.t. $>_{lpo}$ since $>_{lpo}$ contains $\rhd$.

$\underline{plus(s(x), y) >_{lpo} s(plus(x, y))}$, requires a precedence with plus $\sqsupset$ succ

since $plus(s(x), y) >_{lpo} plus(x, y)$,

since $s(x) >_{lpo} x$ and $plus(s(x), y) >_{lpo} y$.

$times(s(x), y) >_{lpo} plus(y, times(x, y))$, requires times $\sqsupset$ plus

since $times(s(x), y) >_{lpo} y$, $\underbrace{times(s(x), y) >_{lpo} times(x, y)}$

since $s(x) >_{lpo} x$,

$times(s(x), y) >_{lpo} y$

So this shows that all rules can be oriented by LPO if one uses a precedence with times $\sqsupset$ plus $\sqsupset$ succ.

To prove termination with LPO:

- start with empty precedence $\sqsupset$.
- Orient one rule after another and extend $\sqsupset$ on demand.
- Whenever $\sqsupset$ is extended, make sure that $\sqsupset$ remains well founded.

Checking whether a TRS can be oriented with some LPO is decidable (since Σ is finite and thus, there are only finitely many possible precedences).
(This is an NP-complete problem that can be implented efficiently using SAT solvers.)

Ex. 4.4.8

$$sum(0, y) \to y$$

$$sum(s(x), y) \to sum(x, s(y))$$

The embedding order fails for the second rule: we have $s(x) \succ_{emb} x$, but $y \not\succ_{emb} s(y)$.

But: $sum(s(x), y) \succ_{lpo} sum(x, s(y))$,

$$\text{Since } s(x) \succ_{lpo} x, \underbrace{sum(s(x), y) \succ_{lpo} s(y)}_{\text{since } sum(s(x),y) \succ_{lpo} y}$$

requires $sum \sqsupset succ$

Now we prove that LPO can indeed be used for termination proofs (i.e., that it is a simplification order).

Thm 4.4.9 (Properties of LPO)

The lexicographic path order is a simplification order.

Proof: We have to prove that LPO

- has subterm property
- is monotonic
- is stable
- is transitive
- is irreflexive

} This implies that LPO is a simpl. order by Thm 4.4.2.

Subterm Property

$f(x_1, \dots, x_i, \dots, x_n) \succ_{lpo} x_i$, since $x_i \succeq_{lpo} x_i$.

Monotonicity

Show that $s \succ_{lpo} t$ implies $q[s]_\pi \succ_{lpo} q[t]_\pi$.

Can easily be proved by structural ind. on π.

Stability

Show that $s \succ_{lpo} t$ implies $s\sigma \succ_{lpo} t\sigma$.

We prove this claim by Noetherian induction with the relation $\rhd^2_{lex}$. ← is well founded by Thm 4.4.5

This means: When proving the claim for (s,t), we can use as ind. hypothesis that it already holds for all (s', t') where $s \rhd s'$ or

$s = s'$ and $t \rhd t'$.

Case analysis according to the def. of LPO.

We have $s \succ_{lpo} t$ and want to show $s\sigma \succ_{lpo} t\sigma$.

<u>Case 1</u>: $s = f(s_1, \dots, s_n)$, $s_i \succeq_{lpo} t$

By ind. hyp: $s_i\sigma \succeq_{lpo} t\sigma$

Thus: $s\sigma = f(s_1\sigma, \dots, s_n\sigma) \succ_{lpo} t\sigma$.

<u>Case 2</u>: $s = f(s_1, \dots, s_n)$, $t = g(t_1, \dots, t_m)$, $f \sqsupset g$, $s \succ_{lpo} t_j$ for all j.

By ind. hyp: $s\sigma \succ_{lpo} t_j\sigma$ for all j.

Thus: $s\sigma = f(s_1\sigma, \dots, s_n\sigma)$, $t\sigma = g(t_1\sigma, \dots, t_m\sigma)$, $f \sqsupset g$, $s\sigma \succ_{lpo} t_j\sigma$ for all j

$\Rightarrow s\sigma \succ_{lpo} t\sigma$

<u>Case 3</u>: $s = f(s_1, \dots, s_i, \dots, s_n)$, $t = f(s_1, \dots, t_i, \dots, t_n)$, $s_i \succ_{lpo} t_i$, $s \succ_{lpo} t_j$ for all j

By ind. hyp.: $s_i\sigma \succ_{lpo} t_i\sigma$, $s\sigma \succ t_j\sigma$ for all j

$\Rightarrow s\sigma \succ_{lpo} t\sigma$

<u>Transitivity</u>: can be proved by induction on $\rhd^3_{lex}$

<u>Irreflexivity</u> We show that $s \not\succ_{lpo} s$ holds by structural induction on s.

<u>Case 1</u>: $s = f(s_1, \dots, s_n)$, $s_i \succeq_{lpo} s$

We also have $s \succ_{lpo} s_i$.

This implies $s_i \succ_{lpo} s_i$ by transitivity. ↯ to the ind. hypothesis

<u>Case 2</u>: $s = f(s_1, \dots, s_n)$: $f \sqsupset f$ would contradict well-foundedness

—

contradict well-foundedness of $\sqsupset$.

Case 3: $s = f(s_1, \dots, s_n)$ with $s_i \succ_{lpo} s_i$ ↯ to the ind. hyp. $\boxtimes$

LPO compares arguments lexicographically from left to right. But one could also compare arguments from right to left (or in any other permutation).

$$pred(O) \to O$$
$$pred(succ(X)) \to X$$
$$minus(X, O) \to X$$
$$minus(X, succ(Y)) \to minus(pred(X), Y)$$

The last rule is not decreasing with $\succ_{lpo}$, because
$X \not\succ_{lpo} pred(X)$

Solution: for minus, the arguments should be compared from right to left. Then

$$minus(X, s(Y)) \succ minus(p(X), Y),$$

since $s(Y) \succ Y$ and $minus(X, s(Y)) \succ p(Y)$ (if $minus \sqsupset pred$)

To make LPO stronger: LPOS (LPO with status):

- every function symbol f of arity n gets a status (a permutation of $1, .., n$)

- when comparing two terms $f(...)$ and $f(...)$, use lexicographic comparison of the arguments where the status of f determines in which order the arguments are compared.
- In the example: minus would need status $\langle 2, 1 \rangle$

first compare the second arguments, then the first

sum would need status $\langle 1, 2 \rangle$